MATH 2040A Tutorial Week 14

§6.8

2. Prove Properties 1-4 on page 423

~~Solution~~ (We only prove, for example, 1 and 3. Others are similar.)

1. If, for any $x \in V$, the functions $L_x, R_x : V \to V$ are defined by
   $L_x(y) = H(x,y)$, $R_x(y) = H(y,x)$ for all $y \in V$,
   then L_x, R_x are linear.

Solution: $L_x(ay_1+by_2) = H(x, ay_1+by_2) = aH(x,y_1)+bH(x,y_2) = aL_x(y_1)+bL_x(y_2)$

R_x — similar

3. For all $x, y, z, w \in V$, $H(x+y, z+w) = H(x,z)+H(x,w)+H(y,z)+H(y,w)$

Solution: $H(x+y, z+w) = H(x, z+w) + H(y, z+w) = H(x,z)+H(x,w)+H(y,z)+H(y,w)$

7. V, W v.s over $F = \mathbb{R}$ or $\mathbb{C}$, $T: V \to W$ a linear map. For any $H \in \mathcal{B}(W)$, define $\hat{T}(H): V \times V \to F$ by $\hat{T}(H)(x,y) = H(T(x), T(y))$ for all $x, y \in V$.

Prove:

a) If $H \in \mathcal{B}(W)$, then $\hat{T}(H) \in \mathcal{B}(V)$

b) $\hat{T}: \mathcal{B}(W) \to \mathcal{B}(V)$ is linear

c) If T is an isomorphism, then so is $\hat{T}$

Proof: a) $\left(\hat{T}(H)\right)(av_1+v_2, v_3) \overset{def}{=} H(T(av_1+v_2), T(v_3)) \overset{T \text{ is linear}}{=} H(aT(v_1)+T(v_2), T(v_3))$

$= aH(T(v_1), T(v_3)) + H(T(v_2), T(v_3)) = a\left(\hat{T}(H)\right)(v_1, v_3) + \left(\hat{T}(H)\right)(v_2, v_3)$

The other equality is proved similarly.

$\Rightarrow \hat{T}(H) \in \mathcal{B}(V)$

b) $\forall H_1, H_2 \in \mathcal{B}(W)$. $\forall v_1, v_2 \in V$

$(\hat{T}(aH_1+bH_2))(v_1, v_2) = (aH_1+bH_2)(Tv_1, Tv_2) = aH_1(Tv_1, Tv_2)+bH_2(Tv_1, Tv_2)$

$= a(\hat{T}(H_1))(v_1, v_2) + b(\hat{T}(H_2))(v_1, v_2) = (a\hat{T}H_1 + b\hat{T}H_2)(v_1, v_2)$

$\Rightarrow \hat{T}$ is linear

c) Note that as ~~a~~ vector spaces over the base field $\mathbb{R}$ or $\mathbb{C}$, $\dim B(V) = \dim B(W)$ ~~Since~~ when $T: V \to W$ is an isomorphism (Why?)

So to prove $\hat{T}: B(W) \to B(V)$ is an isomorphism, we only have to prove that $\hat{T}$ is injective (Why?)

$\hat{T}(H)(v_1, v_2) = 0$ for any $v_1, v_2 \in V$. $\Rightarrow$ $H(T(v_1), T(v_2)) = 0$ for any $v_1, v_2 \in V$

But T is an isomorphism, this implies that $H(w_1, w_2) = 0$ for any $w_1, w_2 \in W$

$\Rightarrow H = 0 \Rightarrow \hat{T}$ is injective

14. V fin. dim v.s. $H \in B(V)$. Prove that, for any ordered basis β, γ of V, $\operatorname{rank}(\psi_\beta(H)) = \operatorname{rand}(\psi_\gamma(H))$

Solution: $\exists$ Q an invertible matrix, s.t.

$$Q^t \psi_\beta(H) Q = \psi_\gamma(H)$$

$$\Rightarrow \operatorname{rank}(\psi_\beta(H)) = \operatorname{rank}(\psi_\gamma(H))$$

16. V v.s. / F, char $F \neq 2$, H symmetric bilinear form on V. Prove that if $K(x) = H(x,x)$ is the quadratic form assoc. with H, then, for all $x, y \in V$,

$$H(x,y) = \frac{1}{2}[K(x+y) - K(x) - K(y)]$$

Solution: $K(x) = H(x,x) \Rightarrow K(x+y) = \underbrace{H(x,x)}_{K(x)} + 2H(x,y) + \underbrace{H(y,y)}_{K(y)}$

~~$\Leftrightarrow$~~ char $F \neq 2 \Rightarrow$ $H(x,y) = \frac{1}{2}(K(x+y) - K(x) - K(y))$

17. K quadratic form on a real inner product space V. Find a symmetric bilinear form H assoc. with K. and an orthonormal basis β for V s.t. $\psi_\beta(H)$ is diagonal

a) $K: R^2 \to R$ $\quad K\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = -2t_1^2 + 4t_1t_2 + t_2^2$

b) $K: R^2 \to R$ $\quad K\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = 7t_1^2 - 8t_1t_2 + t_2^2$

c) $K: R^3 \to R$ $\quad K\begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = 3t_1^2 + 3t_2^2 + 3t_3^2 - 2t_1t_3$

Solution: a) $H\left(\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right) = \frac{1}{2}\left(K\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \end{pmatrix} - K\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - K\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\right)$

$= -2a_1b_1 + 2a_1b_2 + 2a_2b_1 + a_2b_2$

$\Rightarrow \phi_\alpha = \begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}$

Since $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}$

we know that $\beta = \{(1,0), (1,1)\}$

b) $H\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right) = 7a_1b_1 - 4a_1b_2 - 4a_2b_1 + a_2b_2$

$\beta = \{(1,0), (\frac{4}{7}, 1)\}$

c) $H\left(\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}\right) = 3a_1b_1 + 3a_2b_2 + 3a_3b_3 - a_1b_3 - a_3b_1$

$\beta = \{(1,0,0), (0,1,0), (\frac{1}{3}, 0, 1)\}$

24. T linear operator on a real inner product space V. define $H: V \times V \to \mathbb{R}$ by $H(x,y) = \langle x, T(y)\rangle$ for all $x, y \in V$.

a) Prove that H is a bilinear form

b) H is symmetric iff T is self-adjoint

c) When H is an inner product on V

d) H fail to be a bilinear form when V is a complex v.s.

Solution: a) $H(ax_1 + x_2, y) = \langle ax_1 + x_2, T(y)\rangle = a\langle x_1, T(y)\rangle + \langle x_2, T(y)\rangle$

$= aH(x_1, y) + H(x_2, y)$

The other equality in the definition can be proved in the same way.

b) $H(y, x) = \langle y, T(x)\rangle = \langle T(x), y\rangle = \langle x, T^*(y)\rangle$

$H(x,y) = \langle x, T(y)\rangle$

$\Rightarrow$ H is symmetric iff T is self-adjoint.

c) T is positive definite. (see Ex 6.4.22)

(Given inner products $\langle , \rangle'$ $\langle , \rangle$ on V, $\beta = \{v_1, \cdots v_n\}$ is an orthonormal basis for $(V, \langle , \rangle)$. $A = (a_{ij}) = \langle v_j, v_i \rangle'$. Let T be the linear operator s.t. $[T]_\beta = A$, Then, $\langle x, y \rangle' = \langle x, T(y) \rangle$)

d) $H(x, iy) = \langle x, T(iy) \rangle = \langle x, i Ty \rangle = -i \langle x, Ty \rangle \neq i H(x, y)$ (in general)